$$C^m_{1111ii} = c^m_{111} + 2c^m_{112} = -\left[3B^T\left(\frac{\partial c^s_{11}}{\partial P}\right)_T + 3B^T + c^s_{11}\right] = C_a{}^m \quad (36)$$

$$C^m_{1122ii} = 2c^m_{112} + c^m_{123} = -\left[3B^T\left(\frac{\partial c^s_{12}}{\partial P}\right)_T - 3B^T + c^s_{12}\right] = C_b{}^m \quad (37)$$

$$C^m_{1212ii} = c^m_{144} + 2c^m_{166} = -\left[3B^T\left(\frac{\partial c_{44}}{\partial P}\right)_T + 3B^T + c_{44}\right] = C_c{}^m. \quad (38)$$

Alternately, the expressions for $C_a{}^m$, $C_b{}^m$ and $C_c{}^m$ can be obtained directly from the effective elastic constants given earlier by taking the pressure derivatives and evaluating them at zero-pressure. Thus, we find

$$C_a{}^m = -3B^T\left[1 + \left\{\frac{\partial C^s_{11}}{\partial P}\right\}_{P=0}\right] - c^s_{11} \quad (36a)$$

$$C_b{}^m = -3B^T\left[-1 + \left\{\frac{\partial C^s_{12}}{\partial P}\right\}_{P=0}\right] - c^s_{12} \quad (37a)$$

$$C_c{}^m = -3B^T\left[1 + \left\{\frac{\partial C_{44}}{\partial P}\right\}_{P=0}\right] - c_{44}. \quad (38a)$$

To obtain the relation of the second pressure derivatives of the elastic constants to partial contractions of the higher-order elastic constants, we start from equation (34). By taking the pressure derivatives of $(\partial C^s_{ijkl}/\partial P)_T$ and arranging the result,

$$\left(\frac{\partial^2 C^s_{ijkl}}{\partial P^2}\right)_T = \frac{1}{(3B^T)^2}\left[C^s_{ijkl}\left\{1 + 3\left(\frac{\partial B^T}{\partial P}\right)_T\right\} + \left\{4 + 3\left(\frac{\partial B^T}{\partial P}\right)_T\right\}C^m_{ijklmm} + C^m_{ijklmmnn}\right] \quad (39)$$

where $C^m_{ijklmmnn}$ are certain linear combinations of the fourth-order elastic constants and

$$\left(\frac{\partial B^T}{\partial P}\right)_T = -\frac{1}{9B^T}\left[C_a{}^T + 2C_b{}^T\right]. \quad (40)$$

Therefore, solving equation (39) for $C^m_{ijklmmnn}$, we find explicitly the followings:

$$C_a{}^m + 2C_e{}^m = (6B^T - C_a{}^T - 2C_b{}^T)\left[1 + \left(\frac{\partial c^s_{11}}{\partial P}\right)_T\right] + c^s_{11} - 2C_a{}^m + (3B^T)^2\left(\frac{\partial^2 c^s_{11}}{\partial P^2}\right)_T = C_A{}^m \quad (41)$$

$$2C_e{}^m + 3C^m_{1123} = (6B^T - C_a{}^T - 2C_b{}^T)\left[-1 + \left(\frac{\partial c^s_{12}}{\partial P}\right)_T\right] + c^s_{12} - 2C_b{}^m + (3B^T)^2\left(\frac{\partial^2 c^s_{12}}{\partial P^2}\right)_T = C_B{}^m \quad (42)$$

$$C_f{}^m + 2C_g{}^m = (6B^T - C_a{}^T - 2C_b{}^T)\left[1 + \left(\frac{\partial c_{44}}{\partial P}\right)_T\right] + c_{44} - 2C_c{}^m + (3B^T)^2\left(\frac{\partial^2 c_{44}}{\partial P^2}\right)_T = C_C{}^m. \quad (43)$$

Or, from the effective elastic constants given by equations (22–24), we find that

$$C_A{}^m = c^s_{11} - 2C_a{}^m + (6B^T - C_a{}^T - 2C_b{}^T) \times \left[1 + \left\{\frac{\partial C^s_{11}}{\partial P}\right\}_{P=0}\right] + (3B^T)^2\left\{\frac{\partial^2 C^s_{11}}{\partial P^2}\right\}_{P=0} \quad (41a)$$

$$C_B{}^m = c^s_{12} - 2C_b{}^m + (6B^T - C_a{}^T - 2C_b{}^T) \times \left[-1 + \left\{\frac{\partial C^s_{12}}{\partial P}\right\}_{P=0}\right] + (3B^T)^2\left\{\frac{\partial^2 C^s_{12}}{\partial P^2}\right\}_{P=0} \quad (42a)$$

$$C_C{}^m = c_{44} - 2C_c{}^m + (6B^T - C_a{}^T - 2C_b{}^T) \times \left[1 + \left\{\frac{\partial C_{44}}{\partial P}\right\}_{P=0}\right] + (3B^T)^2\left\{\frac{\partial^2 C_{44}}{\partial P^2}\right\}_{P=0}. \quad (43a)$$

Thus, from equations (28–31) and equations (41–43), we have

$$C_A{}^m = c^m_{1111} + 4c^m_{1112} + 2c^m_{1122} + 2c^m_{1123} \quad (44)$$

$$C_B{}^m = 2c^m_{1112} + 2c^m_{1122} + 5c^m_{1123} \quad (45)$$

$$C_C{}^m = c^m_{1144} + 2c^m_{1155} + 4c^m_{1255} + 2c^m_{1266}. \quad (46)$$

These are the primary experimental quantities that are resulting from the ultrasonic-pressure experiments at high pressures.

5. SUMMARY AND CONCLUDING NOTES

Summarizing the foregone sections, the followings may be stated:

(i). Expressions for the effective elastic constants of a cubic crystal subjected to moderately high hydrostatic pressure are derived from the consideration of the rigorous stress-strain relation set by Murnaghan's theory of finite deformations, and these have been compared with the earlier work. Discrepancies were found in the expressions of the effective second-order elastic constants C_{12} and C_{44}. The expressions as ones presented here may be derived from the strain-energy density considerations, and this method may be used to distinguish the observed discrepancies.

(ii). Expressions for the effective second-order elastic constants are

$$C_{\mu\nu} = c_{\mu\nu} + C_I\eta + C_{II}\eta^2$$

where C_I and C_{II} are certain linear combinations of the second- and the higher-order elastic constants of crystal and η is the Lagrangian strain which depends upon pressure. The present expressions are distinguished from the expressions of Birch [1], Seeger and Buck [4], and Thurston [5] and others mainly by the appearance of the $C_{II}\eta^2$ terms in the expressions of the effective elastic constants of cubic crystals.

(iii). Ultrasonic effective second-order elastic constants at high hydrostatic pressures were derived and presented in terms of *thermodynamically mixed* higher-order elastic constants. The rigorous relationships between the pressure derivatives of the elastic constants and partial contractions of the higher-order elastic constants were presented also. And, then, the primary experimental quantities that may be resulting from the ultrasonic-pressure experiments have been identified in a useful form in terms of the *thermodynamically mixed* second-, third-, and fourth-order elastic constants of the crystal under the study.

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